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Recurrences for Alternating Sums of Powers of Binomial Coefficients

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Define $A_r(n)=\sum_{k=-n}^{n}(-1)^k\binom{2n}{n+k}^r$ $(r=2, 3, \ldots)$. An elementary method is given for finding a recurrence for $A_r(n)$ with $[(r+2)/2]$ terms. Using asymptotics it is proved that this is the minimum number of terms in a recurrence for $A_r(n)$ if r is a prime or a power of 2. © 1993 Academic Press, Inc.

1. INTRODUCTION

Define

$$a_r(n)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}^r \qquad (r=2, 3, \ldots).$$

Clearly $a_r(n)=0$ if n is odd and has sign $(-1)^{n/2}$ for even n. For these reasons it is more convenient to work with

$$A_r(n)=\sum_{k=-n}^{n}(-1)^k\binom{2n}{n+k}^r \qquad (r=2, 3, \ldots),$$

which is positive for $n\geqslant 0$. By equating coefficients of x^{2n} in the identity

$$\left[\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}x^k\right]\left[\sum_{k=0}^{2n}\binom{2n}{k}x^k\right]=(1-x)^{2n}(1+x)^{2n}$$
$$=(1-x^2)^{2n}$$
$$=\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}x^{2k}$$

we get

$$A_2(n)=\frac{(2n)!}{(n!)^2}. \tag{1}$$

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In 1903 Dixon [3; 4, 1.2.6, Example 62] proved that

$$\sum_k (-1)^k \binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k} = \frac{(a+b+c)!}{a!\,b!\,c!}.$$

Putting $a=b=c=n$ we get

$$A_3(n) = \frac{(3n)!}{(n!)^3}. \tag{2}$$

One might well expect similar formulae for $A_r(n)$ for larger values of r, but no such formulae are known.

We have the asymptotic formula [1]

$$A_r(n) \sim \frac{\{2\cos(\pi/2r)\}^{2nr+r-1}}{\sqrt{r}\, 2^{r-2}(\pi n)^{(r-1)/2}} \tag{3}$$

for $r \geqslant 2$. The number $\cos^{2r}(\pi/2r)$ is rational if $r=2$ or 3, but if $r>3$ this is no longer true, and therefore $\cos^{2nr}(\pi/2r)$ does not occur in the Stirling formulae for $n!$, $(2n)!$, $(3n)!$, ... Hence we cannot expect simple extensions of Dixon's formula for $r>3$.

Equations (1) and (2) give the recurrences

$$(n+1)A_2(n+1) - 2(2n+1)A_2(n) = 0$$

and

$$(n+1)^2 A_3(n+1) - 3(3n+1)(3n+2)A_3(n) = 0.$$

In 1988 Cusick [2] gave an elementary method for finding a recurrence for

$$S_r(n) = \sum_{k=0}^{n} \binom{n}{k}^r$$

with $[(r+3)/2]$ terms. His method is based on two identities involving the moments

$$S_r(n, t) = \sum_{k=0}^{n} k^t \binom{n}{k}^r \qquad (r \geqslant 1,\ t \geqslant 0).$$

In this paper we develop a modified form of Cusick's method which enables one to find for each r a recurrence for $A_r(n)$ with $[(r+2)/2]$ terms. Using asymptotics we prove that if r is a prime or a power of 2 then the minimum number of terms in a recurrence for $A_r(n)$ is $[(r+2)/2]$. This contrasts with the case for $S_r(n)$, where no lower bounds are known if $r \geqslant 3$. Explicit recurrences for $A_r(n)$ with $r=2, 3, 4, 5, 6$, and 7 are given in the Appendix.

2. The Method

THEOREM 1. *For* $r \geqslant 2$

$$A_r(n) = \sum_{k=-n}^{n} (-1)^k \binom{2n}{n+k}^r$$

satisfies a recurrence with $[(r+2)/2]$ *terms.*

Proof. By the above remarks the theorem is true for $r=2$ and 3. So we can assume that $r \geqslant 4$.

It suffices to show that

$$a_r(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^r$$

satisfies a recurrence of the type

$$P_0(n) a_r(n) + P_1(n) a_r(n-2) + \cdots + P_d(n) a_r(n-2d) = 0, \tag{4}$$

where $d = [r/2]$. Note that $a_r(n) = 0$ for odd n.

Our method is based on two identities which involve the sums

$$a_r(n, t) = \sum_{k=0}^{n} (-1)^k k^t \binom{n}{k}^r \tag{5}$$

where $t \geqslant 0$. Replacing k by $n-k$ in (5) we get the first identity

$$a_r(n, t) = (-1)^n \sum_{j=0}^{t} (-1)^j \binom{t}{j} n^{t-j} a_r(n, j). \tag{6}$$

The second identity is

$$a_r(n, t) = -n^r \sum_{j=0}^{t-r} \binom{t-r}{j} a_r(n-1, j) \qquad (t \geqslant r \geqslant 1), \tag{7}$$

which follows from

$$a_r(n, t) = \sum_{k=1}^{n} (-1)^k n^r k^{t-r} \binom{n-1}{k-1}^r = \sum_{k=0}^{n-1} (-1)^{k+1} n^r (k+1)^{t-r} \binom{n-1}{k}^r.$$

We first supose that $r \geqslant 4$ is even, say $r=2k$, $k \geqslant 2$. We set up $(2k-1)k$ equations in the $(2k-1)k$ variables

$$a_r(n-i, j) \qquad (0 \leqslant i \leqslant 2k-2, k \leqslant j \leqslant 2k-1). \tag{8}$$

First we express all the $a_r(n, j)$ with n even and $1 \leqslant j \leqslant k-1$ in terms of $a_r(n)$ and the variables $a_r(n, j)$ with $k \leqslant j \leqslant 2k-1$. We do this by successively using the $k-1$ identities (6) with $t=1, 3, 5, ..., 2k-3$. Thus with $t=1$ in (6) we get $a_r(n, 1)=na_r(n)/2$, with $t=3$ we get $a_r(n, 2)=(n^3a_r(n)+4a_r(n, 3))/6n$, and so on. Cusick [2] refers to this procedure as "pushing up the subscript j," We express all the $a_r(n, j)$ with n odd and $0 \leqslant j \leqslant k-1$ in terms of the variables $a_r(n, j)$ with $k \leqslant j \leqslant 2k-1$. We do this by successively using the k identities (6) with $t=0, 2, 4, .., 2k-2$. Thus with $t=0$ in (6) we get $a_r(n)=0$, with $t=2$ we get $a_r(n, 1)=a_r(n, 2)/n$, and so on.

By the above remarks, identity (6) with n even and $t=2k-1$ gives an equation which involves only the sum $a_r(n)$ and the variables $a_r(n, j)$ with $k \leqslant j \leqslant 2k-1$. Let $f_1(n)$ denote this equation.

Identity (6) with n even and $t=2k+1$ gives an equation which (after pushing up the subscript j) involves only the variables $a_r(n, j)$ with $k \leqslant j \leqslant 2k-1$ and the sums $a_r(n)$, $a_r(n, 2k)$ and $a_r(n, 2k+1)$. Using identities (7) with $t=2k$ and $t=2k+1$, and by pushing up the subscript j, we can replace the sums $a_r(n, 2k)$ and $a_r(n, 2k+1)$ by expressions involving only the variables $a_r(n-1, j)$ with $k \leqslant j \leqslant 2k-1$. The result is an equation which involves only $a_r(n)$ and the $a_r(n-i, j)$ with $i=0, 1$ and $k \leqslant j \leqslant 2k-1$. Let $f_2(n)$ denote this equation.

Equations $f_m(n)$ with n even and $3 \leqslant m \leqslant k+1$ are obtained in a similar manner. We start with identity (6) with n even and $t=2k+2m-3$. After pushing up the subscript j this gives an equation which involves only the variables $a_r(n, j)$ with $k \leqslant j \leqslant 2k-1$ and the sums $a_r(n)$ and $a_r(n, j)$ with $2k \leqslant j \leqslant 2k+2m-3$. We then use identities (7) with $2k \leqslant t \leqslant 2k+2m-3$ and the process of pushing up the subscript j to replace the $a_r(n, j)$ with $2k \leqslant j \leqslant 2k+2m-3$ by expressions involving only the $a_r(n-1, j)$ with $k \leqslant j \leqslant 2k-1$. The result is an equation $f_m(n)$ which involves only $a_r(n)$ and the $a_r(n-i, j)$ with $i=0, 1$ and $k \leqslant j \leqslant 2k-1$.

Now we obtain equations $g_m(n)$ with n odd and $1 \leqslant m \leqslant k$. We start with identity (6) with n odd and $t=2k+2m-2$. By pushing up the subscript j we obtain an equation which involves only the $a_r(n, j)$ with $k \leqslant j \leqslant 2k-1$ and the $a_r(n, j)$ with $2k \leqslant j \leqslant 2k+2m-2$. We then use identities (7) with $2k \leqslant t \leqslant 2k+2m-2$ and the process of pushing up the subscript j to replace the $a_r(n, j)$ with $2k \leqslant j \leqslant 2k+2m-2$ by expressions involving only $a_r(n-1)$ and the $a_r(n-1, j)$ with $k \leqslant j \leqslant 2k-1$. The result is an equation $g_m(n)$ which involves only $a_r(n-1)$ and the $a_r(n-i, j)$ with $i=0, 1$ and $k \leqslant j \leqslant 2k-1$.

We consider the system of $(2k-1)k$ equations $f_1(n-2i)$, $f_2(n-2i)$, ..., $f_{k+1}(n-2i)$, $g_1(n-2i-1)$, $g_2(n-2i-1)$, ..., $g_k(n-2i-1)$, and $f_1(n-2k+2)$, where n is even and $0 \leqslant i \leqslant k-2$. This system of $(2k-1)k$ equations involves only the sums $a_r(n-2i)$ with $0 \leqslant i \leqslant k-1$ and the $(2k-1)k$ variables (8). The system is obviously consistent, so we can solve it uniquely for the variables (8) in terms of the $a_r(n-2i)$ with $0 \leqslant i \leqslant k-1$, provided that the determinant of the system (which is a polynomial in n) is not identically 0. Even if this determinant is 0, we can still solve the system by appropriate choice of those variables in (8) which are available as free parameters. Given a solution of the system, there are many ways to obtain a recurrence of the type (4). For example, we can replace n by $n+2$ in the expression obtained for $a_r(n-2, k)$ and subtract this from the expression obtained for $a_r(n, k)$.

It remains to consider the case when $r \geqslant 5$ is odd, say $r = 2k+1$, $k \geqslant 2$. This is very similar to the case when r is even. We set up $(2k-1)k$ equations in the $(2k-1)k$ variables

$$a_r(n-i, j) \qquad (0 \leqslant i \leqslant 2k-2,\ k+1 \leqslant j \leqslant 2k). \tag{9}$$

First we express all the $a_r(n, j)$ with n even and $1 \leqslant j \leqslant k$ in terms of $a_r(n)$ and $a_r(n, j)$ with $k+1 \leqslant j \leqslant 2k$. We push up the subscript j by successively using the k identities (6) with $t = 1, 3, 5, ..., 2k-1$. We express all the $a_r(n, j)$ with n odd and $0 \leqslant j \leqslant k-1$ in terms of the $a_r(n, j)$ with $k+1 \leqslant j \leqslant 2k$. We do this by successively using the k identities (6) with $t = 0, 2, 4, ..., 2k$.

Equations $f_m(n)$ with n even and $1 \leqslant m \leqslant k+1$ are obtained in the following manner. We start with identity (6) with n even and $t = 2k+2m-1$. After pushing up the subscript j we obtain an equation which involves only the variables $a_r(n, j)$ with $k+1 \leqslant j \leqslant 2k$ and the sums $a_r(n)$ and $a_r(n, j)$ with $2k+1 \leqslant j \leqslant 2k+2m-1$. We then use identities (7) with $2k+1 \leqslant t \leqslant 2k+2m-1$ and the process of pushing up the subscript j to replace the $a_r(n, j)$ with $2k+1 \leqslant j \leqslant 2k+2m-1$ by expressions involving only the $a_r(n-1, j)$ with $k+1 \leqslant j \leqslant 2k$. The result is an equation $f_m(n)$ which involves only $a_r(n)$ and the $a_r(n-i, j)$ with $i = 0, 1$ and $k+1 \leqslant j \leqslant 2k$. Note that equation $f_1(n)$ involves only the sum $a_r(n)$ and the variables $a_r(n, j)$ with $k+1 \leqslant j \leqslant 2k$. This is because $a_r(n, 2k+1) = a_r(n, r) = -n^r a_r(n-1) = 0$ for even n.

Now we obtain equations $g_m(n)$ with n odd and $1 \leqslant m \leqslant k$. We start with identity (6) with n odd and $t = 2k+2m$. By pushing up the subscript j we obtain an equation which involves only the $a_r(n, j)$ with $k+1 \leqslant j \leqslant 2k$ and the $a_r(n, j)$ with $2k+1 \leqslant j \leqslant 2k+2m$. We then use identities (7) with $2k+1 \leqslant t \leqslant 2k+2m$ and the process of pushing up the subscript j to replace the $a_r(n, j)$ with $2k+1 \leqslant j \leqslant 2k+2m$ by expressions involving only

$a_r(n-1)$ and the $a_r(n-1, j)$ with $k+1 \leqslant j \leqslant 2k$. The result is an equation $g_m(n)$ which involves only $a_r(n-1)$ and the $a_r(n-i, j)$ with $i=0, 1$ and $k+1 \leqslant j \leqslant 2k$.

We consider the system of $(2k-1)k$ equations $f_1(n-2i)$, $f_2(n-2i)$, ..., $f_{k+1}(n-2i)$, $g_1(n-2i-1)$, $g_2(n-2i-1)$, ..., $g_k(n-2i-1)$, and $f_1(n-2k+2)$, where n is even and $0 \leqslant i \leqslant k-2$. This system involves only the $a_r(n-2i)$ with $0 \leqslant i \leqslant k-1$ and the $(2k-1)k$ variables (9). Recall that by the above remarks equation $f_1(n-2k+2)$ involves only $a_r(n-2k+2)$ and the variables $a_r(n+2k+2, j)$ with $k+1 \leqslant j \leqslant 2k$. This system of $(2k-1)k$ equations is obviously consistent, so we can solve it for the variables (9) in terms of the sums $a_r(n-2i)$ with $0 \leqslant i \leqslant k-1$. Given a solution of the system, we proceed as in the case of n even to find a recurrence of the type (4) for $a_r(n)$. There are many ways to do this, but all involve replacing n by $n+2$. ▮

Remark 1. It seems very unlikely that the determinant of our system of $(2k-1)k$ equations is zero; this would give a recurrence for $A_r(n)$ with fewer than $[(r+2)/2]$ terms. It may be that a detailed examination of the system would yield a proof that its determinant is nonzero.

Remark 2. The method described above applies in a degenerate form to the cases $r=2$ and 3.

3. The Use of Asymptotics

Let $F: \mathbf{Z} \to \mathbf{Z}$ be such that $F(n) \neq 0$ for n sufficiently large. Suppose $F(n)$ satisfies

$$P_0(n)F(n) + P_1(n)F(n-1) + \cdots + P_d(n)F(n-d) = 0 \tag{10}$$

for all n, where

$$P_i(n) = c_{i0} + c_{i1}n + \cdots + c_{ir}n^r.$$

Here r denotes the maximum degree of $P_i(n)$ for $0 \leqslant i \leqslant d$. Now suppose that

$$\lim_{n \to \infty} \frac{F(n+1)}{F(n)} = \mu.$$

Then $\lim_{n \to \infty} F(n-i)/F(n-d) = \mu^{d-i}$ for $0 \leqslant i \leqslant d$. Hence, dividing (10) by $n^r F(n-d)$ and letting $n \to \infty$, we get

$$0 = \lim_{n \to \infty} \sum_{i=0}^{d} \frac{P_i(n)}{n^r} \frac{F(n-i)}{F(n-d)} = \sum_{i=0}^{d} c_{ir}\mu^{d-i} = p(\mu),$$

where $p(t) = c_{0r}t^d + c_{1r}t^{d-1} + \cdots + c_{dr}$ is the characteristic polynomial for recurrence (10). This implies that μ is algebraic over $\mathbf{Q}$ with degree at most d. In particular if $\deg_{\mathbf{Q}} \mu = d$, then $F(n)$ cannot satisfy a recurrence with fewer than $d+1$ terms.

4. A Minimality Proof

THEOREM 2. *If r is a prime or a power of 2, then the minimum number of terms in a recurrence for*

$$A_r(n) = \sum_{k=-n}^{n} (-1)^k \binom{2n}{n+k}^r$$

is $[(r+2)/2]$.

Proof. From the asymptotic formula (3) we get

$$\mu = \lim_{n \to \infty} \frac{A_r(n+1)}{A_r(n)} = \{2\cos(\pi/2r)\}^{2r}.$$

Therefore by the above argument, it suffices to show that

$$\deg_{\mathbf{Q}} \{\cos^{2r}(\pi/2r)\} = [r/2]$$

whenever r is a prime or a power of 2. Since $[r/2] = \frac{1}{2}\phi(2r)$ if and only if r is a prime or a power of 2, where $\phi(n)$ is the Euler totient function, it is enough to show that

$$\deg_{\mathbf{Q}} \{\cos^{2r}(\pi/2r)\} = \tfrac{1}{2}\phi(2r)$$

for $r \geqslant 2$. Recall that $\cos(2\pi/4r) = (e^{2\pi i/4r} + e^{-2\pi i/4r})/2$. Since $\cos^{2r}(2\pi/4r)$ is invariant under exactly four automorphisms ($e^{2\pi i/4r} \mapsto \pm e^{\pm 2\pi i/4r}$) and $\deg_{\mathbf{Q}} e^{2\pi i/4r} = \phi(4r)$, we have

$$\deg_{\mathbf{Q}} \{\cos^{2r}(\pi/2r)\} = \deg_{\mathbf{Q}} \{\cos^{2r}(2\pi/4r)\} = \tfrac{1}{4}\phi(4r) = \tfrac{1}{2}\phi(2r)$$

for $r \geqslant 2$, which completes the proof. ∎

Appendix

Recurrences for

$$A_r(n) = \sum_{k=-n}^{n} (-1)^k \binom{2n}{n+k}^r$$

for $r = 2, 3, 4, 5, 6$, and 7 are written in the form

$$\sum_{i=0}^{[r/2]} P_i(n) A_r(n+1-i) = 0.$$

$r=2$:

$$P_0(n)=n+1,$$
$$P_1(n)=-2(2n+1).$$

$r=3$:

$$P_0(n)=(n+1)^2,$$
$$P_1(n)=-3(3n+1)(3n+2).$$

$r=4$:

$$P_0(n)=(n+1)^3\,(2n+1)(48n^2-30n+5),$$
$$P_1(n)=-2(6528n^6+8976n^5+2000n^4-1606n^3-513n^2+95n+35),$$
$$P_2(n)=4n(2n-1)^3\,(48n^2+66n+23).$$

$r=5$:

$$\begin{aligned}P_0(n)&=(n+1)^4\,(2n+1)^2\,(220n^3-198n^2+63n-7),\\ P_1(n)&=-5(110{,}000n^9+231{,}000n^8+140{,}400n^7-11{,}890n^6\\ &\qquad -35{,}415n^2-4235n^4+3828n^3+795n^2-161n-42),\\ P_2(n)&=5(2n-1)^2\,(5n-3)(5n-1)(5n+1)\\ &\qquad \times(220n^3+462n^2+327n+78).\end{aligned}$$

$r=6$:

$$\begin{aligned}P_0(n)=n(n+1)^5\,(2n+1)^3\,&(78{,}037{,}440n^{10}-617{,}796{,}400n^9\\ &+2{,}125{,}175{,}000n^8-4{,}169{,}616{,}100n^7+5{,}152{,}323{,}982n^6\\ &-4{,}181{,}430{,}032n^5+2{,}256{,}662{,}768n^4\\ &-801{,}756{,}137n^3+180{,}454{,}862n^2-23{,}380{,}182n+1{,}331{,}694),\\ P_1(n)=-n(&1{,}726{,}812{,}472{,}320n^{18}-6{,}763{,}348{,}849{,}920n^{17}\\ &+4{,}315{,}929{,}463{,}040n^{16}+12{,}820{,}329{,}932{,}800n^{15}\\ &-14{,}935{,}739{,}835{,}744n^{14}-9{,}993{,}368{,}296{,}496n^{13}\\ &+16{,}064{,}720{,}930{,}016n^{12}+4{,}247{,}936{,}247{,}656n^{11}\\ &-8{,}757{,}710{,}219{,}082n^{10}-1{,}162{,}565{,}196{,}494n^9\\ &+2{,}648{,}485{,}238{,}876n^8+227{,}381{,}394{,}095n^7\\ &-455{,}383{,}606{,}899n^6-27{,}261{,}740{,}519n^5+48{,}750{,}795{,}445n^4\\ &+1{,}840{,}189{,}750n^3-3{,}014{,}020{,}100n^2-53{,}955{,}972n+82{,}565{,}028),\end{aligned}$$

$$P_2(n) = (2n-1)(53{,}979{,}745{,}847{,}040n^{18} - 319{,}380{,}162{,}928{,}320n^{17}$$
$$+ 645{,}960{,}236{,}561{,}120n^{16} - 255{,}536{,}583{,}113{,}760n^{15}$$
$$- 876{,}132{,}941{,}121{,}568n^{14} + 1{,}131{,}210{,}134{,}867{,}332n^{13}$$
$$+ 77{,}277{,}359{,}111{,}190n^{12} - 917{,}344{,}636{,}358{,}080n^{11}$$
$$+ 400{,}356{,}174{,}225{,}626n^{10} + 257{,}655{,}918{,}764{,}339n^{9}$$
$$- 238{,}160{,}804{,}963{,}692n^{8} + 7{,}795{,}085{,}753{,}245n^{7}$$
$$+ 46{,}175{,}053{,}301{,}278n^{6} - 13{,}381{,}382{,}620{,}737n^{5}$$
$$- 1{,}396{,}337{,}420{,}016n^{4} + 993{,}296{,}610{,}621n^{3}$$
$$- 36{,}478{,}231{,}398n^{3} - 29{,}126{,}737{,}080n + 3{,}353{,}203{,}980),$$

$$P_3(n) = -8(n-1)^3\,(2n-3)^5\,(2n-1)(78{,}037{,}440n^{10} + 162{,}578{,}000n^{9}$$
$$+ 76{,}692{,}200n^{8} - 44{,}393{,}700n^{7} - 37{,}123{,}918n^{6} + 3{,}464{,}240n^{5}$$
$$+ 5{,}574{,}838n^{4} - 94{,}045n^{3} - 301{,}231n^{2} + 35n + 6895).$$

$r = 7$:

$$P_0(n) = n^2(n+1)^6\,(2n+1)^4\,(6{,}131{,}808{,}256n^{14} - 70{,}515{,}794{,}944n^{13}$$
$$+ 368{,}559{,}403{,}520n^{12} - 1{,}158{,}709{,}696{,}256n^{11}$$
$$+ 2{,}444{,}438{,}502{,}236n^{10} - 3{,}655{,}501{,}382{,}688n^{9}$$
$$+ 3{,}991{,}427{,}118{,}416n^{8} - 3{,}229{,}981{,}644{,}344n^{7}$$
$$+ 1{,}946{,}034{,}566{,}176n^{6} - 869{,}073{,}329{,}368n^{5}$$
$$+ 283{,}556{,}917{,}800n^{4} - 65{,}679{,}912{,}264n^{3}$$
$$+ 10{,}235{,}033{,}883n^{2} - 962{,}950{,}329n + 41{,}369{,}130),$$

$$P_1(n) = -7n^2(168{,}256{,}818{,}544{,}640n^{24} - 1{,}093{,}669{,}320{,}540{,}160n^{23}$$
$$+ 2{,}319{,}545{,}521{,}348{,}608n^{22} - 380{,}676{,}119{,}957{,}504n^{21}$$
$$- 5{,}215{,}588{,}134{,}335{,}488n^{20} + 5{,}323{,}663{,}295{,}881{,}728n^{19}$$
$$+ 3{,}836{,}808{,}002{,}659{,}072n^{18} - 7{,}875{,}464{,}989{,}436{,}800n^{17}$$
$$- 361{,}585{,}251{,}246{,}912n^{16} + 5{,}803{,}483{,}695{,}985{,}696n^{15}$$
$$- 1{,}154{,}926{,}572{,}124{,}048n^{14} - 2{,}565{,}991{,}757{,}401{,}944n^{13}$$
$$+ 812{,}144{,}115{,}509{,}248n^{12} + 727{,}743{,}204{,}301{,}256n^{11}$$
$$- 269{,}479{,}235{,}856{,}528n^{10} - 137{,}487{,}239{,}706{,}632n^{9}$$

$$+ 53{,}473{,}260{,}623{,}129n^8 + 17{,}718{,}664{,}093{,}092n^7$$
$$- 7{,}112{,}348{,}485{,}615n^6 - 1{,}502{,}878{,}803{,}768n^5 + 618{,}673{,}874{,}223n^4$$
$$+ 75{,}941{,}186{,}148n^3 - 31{,}945{,}013{,}469n^2 - 1{,}736{,}943{,}912n$$
$$+ 744{,}644{,}340),$$

$$\begin{aligned}
P_2(n) = 7(&84{,}094{,}747{,}908{,}611{,}072n^{26} - 798{,}900{,}200{,}131{,}805{,}184n^{25}\\
&+ 3{,}123{,}478{,}378{,}805{,}485{,}568n^{24} - 6{,}013{,}067{,}405{,}372{,}805{,}120n^{23}\\
&+ 4{,}080{,}153{,}273{,}930{,}216{,}448n^{22} + 5{,}773{,}603{,}531{,}081{,}921{,}024n^{21}\\
&- 14{,}002{,}768{,}368{,}629{,}026{,}560n^{20} + 7{,}506{,}981{,}003{,}066{,}591{,}872n^{19}\\
&+ 7{,}935{,}866{,}814{,}359{,}521{,}856n^{18} - 12{,}705{,}859{,}453{,}477{,}230{,}432n^{17}\\
&+ 2{,}983{,}042{,}317{,}348{,}989{,}392n^{16} + 5{,}808{,}776{,}305{,}832{,}389{,}000n^{15}\\
&- 4{,}664{,}595{,}244{,}719{,}102{,}000n^{14} - 108{,}280{,}281{,}655{,}488{,}160n^{13}\\
&+ 1{,}656{,}281{,}654{,}344{,}289{,}552n^{12} - 656{,}256{,}882{,}546{,}486{,}552n^{11}\\
&- 129{,}369{,}922{,}527{,}885{,}761n^{10} + 164{,}199{,}072{,}616{,}514{,}851n^9\\
&- 33{,}245{,}064{,}021{,}597{,}093n^8 - 7{,}797{,}309{,}352{,}921{,}897n^7\\
&+ 3{,}889{,}858{,}807{,}200{,}273n^6 - 45{,}171{,}048{,}843{,}963n^5\\
&- 214{,}433{,}803{,}776{,}879n^4 + 30{,}335{,}687{,}605{,}557n^3\\
&+ 3{,}673{,}614{,}465{,}972n^2 - 1{,}214{,}808{,}402{,}276n + 83{,}469{,}566{,}880),
\end{aligned}$$

$$\begin{aligned}
P_3(n) = &-7(n-1)^2(2n-3)^4(7n-11)(7n-10)(7n-9)\\
&\times(7n-8)(7n-6)(7n-5)\\
&\times(6{,}131{,}808{,}256n^{14} + 15{,}329{,}520{,}640n^{13} + 9{,}848{,}620{,}544n^{12}\\
&- 4{,}250{,}654{,}464n^{11} - 6{,}024{,}813{,}888n^{10} + 5{,}864{,}544n^9\\
&+ 1{,}444{,}239{,}344n^8 + 115{,}440{,}584n^7 - 183{,}032{,}744n^6\\
&- 13{,}509{,}104n^5 + 12{,}429{,}768n^4 + 745{,}920n^3\\
&- 500{,}469n^2 - 17{,}115n + 9324).
\end{aligned}$$

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